

Inverse energy cascade in three-dimensional isotropic turbulence

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In turbulent flows kinetic energy is spread by nonlinear interactions over a broad range of scales. Energy transfer may proceed either toward small scales or in the reverse direction. The latter case is peculiar of two-dimensional (2D) flows. Interestingly, a reversal of the energy flux is observed also in three-dimensional (3D) geophysical flows under rotation and/or confined in thin layers. The question is whether this phenomenon is enforced solely by external anisotropic mechanisms or it is intimately embedded in the Navier-Stokes (NS) equations. Here we show that an inverse energy cascade occurs also in 3D isotropic flow. The flow is obtained from a suitable surgery of the NS equations, keeping only triadic interactions between sign-defined helical modes, preserving homogeneity and isotropy and breaking reflection invariance. Our findings highlight the role played by helicity in the energy transfer process and show that both 2D and 3D properties naturally coexist in all flows in nature.

Inviscid invariants of the NS equations are crucial in determining the direction of the turbulent energy transfer [1]. In some cases, as for fully isotropic and homogeneous turbulence in 2D the presence of two positive-defined invariants (energy and enstrophy) does not allow a stationary transfer of both quantities, neither to small nor to large scales [2]. In presence of two fluxes, they must necessarily flow in opposite directions [3–7] and this remains true even for turbulent systems in non-integer dimensions obtained by fractal Fourier decimation [8]. The fluid equations possess two inviscid invariants also in 3D: energy and helicity (i.e. the scalar product of velocity and vorticity). The inviscid conservation of helicity was discovered relatively recently [16, 17]. At variance with energy, helicity is not positive defined. This allows for a simultaneous small-scale transfer of energy and helicity, as confirmed by results of two-point closures [17–19] and direct numerical simulations [20, 21]. Nevertheless, a reversal of the flux of energy has been observed in geophysical flows subject to earth rotation [9, 10] as well as in shallow fluid layers [11–15]. In both cases, this phenomenon is accompanied by strong anisotropic effects and by a substantial two-dimensionalization of the flow, induced either by the rotation or by the effects of confinement. Moreover, rotations injects fluctuations in the helical sector while a perfect two-dimensional flow has vanishing *point-wise* helicity, being vorticity always orthogonal to velocity. The role played by helicity in the energy transfer mechanism of 3D flows has attracted a broad scientific interest (see, e.g., [21] and reference therein). Dynamical systems have been developed to study in details energy and helicity transfer at high Reynolds numbers [22, 23]. Further, speculation connecting the existence of intermittent burst in the energy cascade induced by “local” helicity blocking mechanism have been proposed [22]. Despite these important contributions, the understanding of the phenomenology of helicity remains “mysterious”,

as summarized in the conclusion of a recent state-of-the-art numerical study [21]. Here we present theoretical and numerical evidences of a new phenomenon induced by helicity conservation: a statistically stationary *backward* energy transfer can be sustained even in 3D fully isotropic turbulence.

The starting point of our analysis is the well-known helical decomposition [19] of the velocity field $\mathbf{v}(\mathbf{x})$, expanded in Fourier series, $\mathbf{u}(\mathbf{k})$:

$$\mathbf{u}(\mathbf{k}) = u^+(\mathbf{k})\mathbf{h}^+(\mathbf{k}) + u^-(\mathbf{k})\mathbf{h}^-(\mathbf{k}) \quad (1)$$

where \mathbf{h}^\pm are the eigenvectors of the curl operator $i\mathbf{k} \times \mathbf{h}^\pm = \pm k\mathbf{h}^\pm$. In particular, we choose $\mathbf{h}^\pm = \hat{\mathbf{v}} \times \hat{\mathbf{k}} \pm i\hat{\mathbf{v}}$, where $\hat{\mathbf{v}}$ is an arbitrary versor orthogonal to \mathbf{k} which satisfies the relation $\hat{\mathbf{v}}(\mathbf{k}) = -\hat{\mathbf{v}}(-\mathbf{k})$ (necessary to ensure the reality of the velocity field). Such requirement is satisfied e.g. by the choice $\hat{\mathbf{v}} = \mathbf{z} \times \mathbf{k}/\|\mathbf{z} \times \mathbf{k}\|$, with \mathbf{z} an arbitrary vector. In terms of this *exact* decomposition of each Fourier mode energy, $E = \int d^3x |\mathbf{v}(\mathbf{x})|^2$, and helicity, $H = \int d^3x \mathbf{v} \cdot \mathbf{w}$, where \mathbf{w} is the vorticity, are written as:

$$\begin{cases} E = \sum_{\mathbf{k}} |u^+(\mathbf{k})|^2 + |u^-(\mathbf{k})|^2; \\ H = \sum_{\mathbf{k}} k(|u^+(\mathbf{k})|^2 - |u^-(\mathbf{k})|^2). \end{cases} \quad (2)$$

Similarly, the non-linear term of the NS equations can be exactly decomposed in 4 independent classes of triadic interactions, determined by the helical content of the complex amplitudes, $\mathbf{u}^{s_k}(\mathbf{k})$ with $s_k = \pm$ (see [19]). Among three generic interacting modes $\mathbf{u}^{s_k}(\mathbf{k}), \mathbf{u}^{s_p}(\mathbf{p}), \mathbf{u}^{s_q}(\mathbf{q})$, one can identify 8 different helical combinations ($s_k = \pm, s_p = \pm, s_q = \pm$). Among them, only four are independent because of the symmetry that allows to change all signs of helicity simultaneously. Let us now consider the dynamics of an incompressible flow $\nabla \cdot \mathbf{v} = 0$, which is determined by a decimated NS equation in which all interactions between modes have been switched off except for those with a well defined sign of helicity, e.g. positive

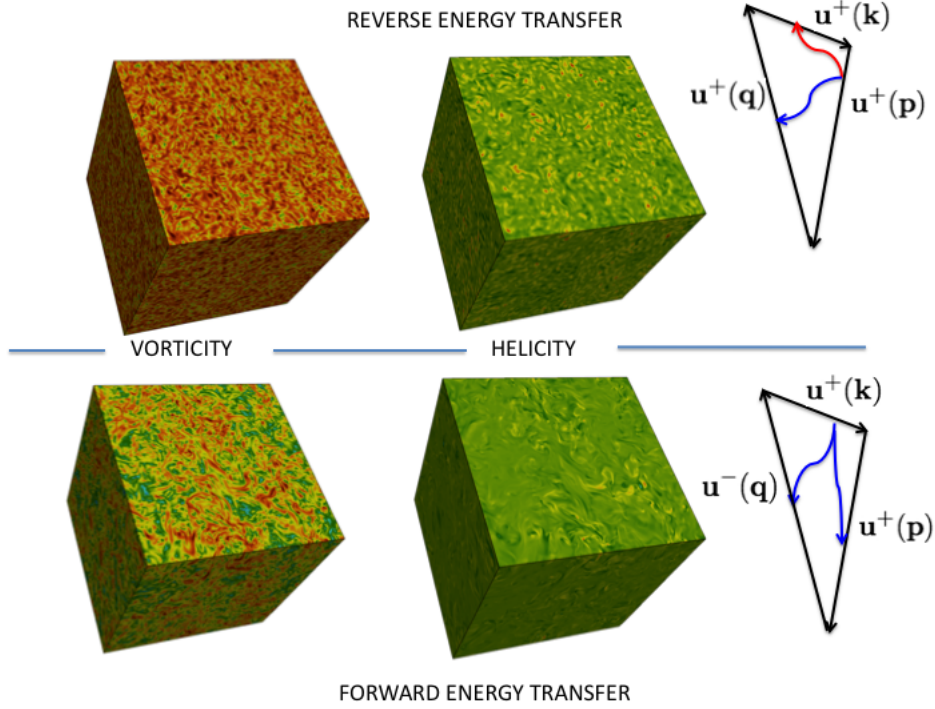


FIG. 1: Comparison between helicity and vorticity fields for normal NS turbulence (bottom row) and for inverse cascade 3D turbulence (top row). We also represent a pictorial scheme of triads responsible for the inverse cascade regime as suggested in [19]. Inverse 3D cascade is possible with triads where all helical components have the same sign. In this case the middle wave number (here $u^+(\mathbf{p})$) transfers energy also to smaller wavenumbers ($u^+(\mathbf{k})$), at difference from what happens for triads with different helical components where it is always the smallest wavenumber $u^+(\mathbf{k})$ that transfer energy to higher modes $u^-(\mathbf{q}), u^+(\mathbf{p})$.

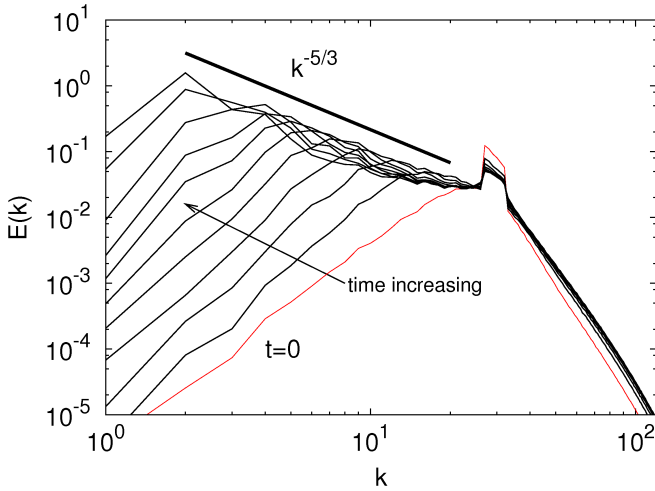


FIG. 2: (color online) Non-stationary spectrum in the inverse energy cascade regime. Red, dashed line represent $k^{-5/3}$ slope.

($s_k = +, s_p = +, s_q = +$). We define the projector on

positive/negative helicity states as

$$\mathcal{P}^\pm \equiv \frac{\mathbf{h}^\pm \otimes \overline{\mathbf{h}^\pm}}{\mathbf{h}^\pm \cdot \mathbf{h}^\pm} \quad (3)$$

where $\bar{\cdot}$ stands for complex conjugate. Then we project the velocity field into its positive helicity component:

$$\mathbf{v}^+(\mathbf{x}) \equiv \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \mathcal{P}^+ \mathbf{u}(\mathbf{k}); \quad (4)$$

and we consider the decimated NS equations:

$$\partial_t \mathbf{v}^+ = (-\mathbf{v}^+ \cdot \nabla \mathbf{v}^+ - \nabla p)^+ + \nu \Delta \mathbf{v}^+ + \mathbf{f}^+ \quad (5)$$

where ν is the viscosity, p is the pressure and \mathbf{f} is the external forcing stirring the fluid around a wavevector k_f . The non-linear term and the forcing are projected on the positive helicity states with the same procedure followed for the velocity field (4). The resulting system has two positive definite invariants, see Eq. (2), the energy and the helicity, $H = \sum_{\mathbf{k}} k |\mathbf{u}^+(\mathbf{k})|^2$, and contains only interactions between positive helicity modes. Helicity becomes a coercitive quantity: the decimated NS equations cannot sustain a simultaneous forward cascade of energy and helicity, for the same arguments which forbid the existence of a simultaneous forward cascade of energy

and enstrophy in 2D turbulence [2, 19]. Therefore, the dynamics of Eq. (5) should display a double cascade phenomenology, characterized by an inverse energy cascade with Kolmogorov spectrum $E(k) \sim k^{-5/3}$ for $k \ll k_f$, and a direct helicity cascade with a $k^{-7/3}$ spectrum for $k \gg k_f$. It is interesting to note that, at variance with usual 3D NS dynamics, such flow should not display dissipative anomaly for kinetic energy, i.e. energy dissipation should vanish in the limit $\nu \rightarrow 0$. Indeed, the direct helicity cascade carries also a residual, non-constant flux of kinetic energy toward small scales which decays as k^{-1} and therefore vanishes in the high Reynolds number limit. As a consequence, one may speculate that the decimated NS equations possess a less singular spatio-temporal evolution. Numerical simulations have been performed with a fully-dealiased, pseudo-spectral code at resolution 512^3 on a triply periodic cubic domain of size $L = 2\pi$. The flow is sustained by a random Gaussian forcing, with $\langle f_i(\mathbf{k}, t) f_j(\mathbf{q}, t') \rangle = F(k) \delta(\mathbf{k} - \mathbf{q}) \delta(t - t') Q_{ij}(\mathbf{k})$, where $Q_{ij}(\mathbf{k})$ is a projector assuring incompressibility and $F(k)$ has support only in the high wavenumber range $|k| \in [k_{min} = 25 : k_{max} = 32]$.

A visual inspection of the helicity and vorticity fields offered in Fig. 1 shows the differences between the forward cascade, which develops in standard 3D NS equations forced at large scale, and the novel 3D inverse cascade regime obtained from the decimated NS equation (5) forced at small scales (see Fig. 1). The latter does not possess any filamentary structure in the vorticity field, witnessing the fact that the vortex stretching mechanism, which is responsible for the forward cascade in standard 3D systems, is here reduced. Similarly, the forward regime does not possess any coherent helicity signal at variance with the inverse regime which shows strong non-homogeneity in the helical spatial distribution.

In Fig. 2 we show a typical evolution of the energy spectrum obtained from Eq. (5) by initializing the flow with energy only at high wave-numbers. The development of an inverse cascade with a Kolmogorov spectrum $E(k) \sim k^{-5/3}$ is unambiguous.

In absence of a large-scale dissipative mechanism, the inverse cascade would accumulate the kinetic energy in the lowest available mode, originating a *condensed state* [14]. In order to avoid this phenomenon we made a second series of numerical simulations, adding an hypoviscosity at large scale $\propto \Delta^{-1}v$. In such a case, the total kinetic energy becomes stationary as shown in Fig. 3, and is equally distributed among the three velocity components, showing that the flow is fully isotropic. This allows to study scaling properties without having to cope with anisotropic sub-leading contributions [26]. In the inset of Fig. 3, we show the stationary energy flux in Fourier space, defined as $\Pi(k) \equiv (d/dt) \int_k^\infty E(p) dp$ where time derivative is computed by taking into account only the non-linear terms of Eq. (5). The negative plateaux in the

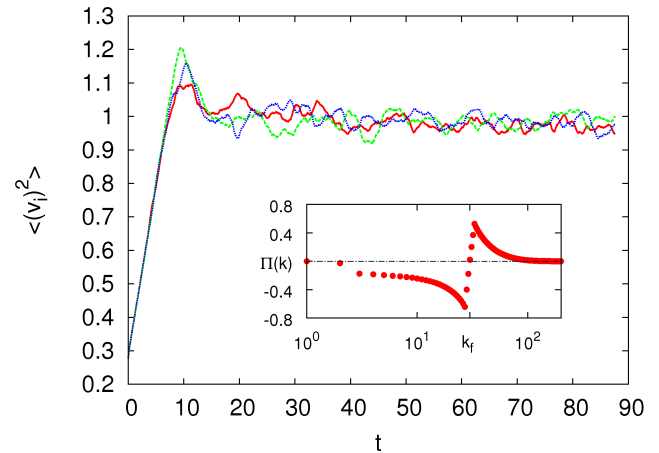


FIG. 3: (color online) Evolution of the three components of the turbulent kinetic energy as a function of time, $\langle (v_i)^2 \rangle$, with $i = x$ (red, solid line) $i = y$ (green, dashed line) and $i = z$ (blue, dotted line). Inset: energy flux, $\Pi(k)$, in the Fourier space. Notice the clear negative plateaux in the inertial range $k < k_f$.

inertial range of wave-numbers is a clear indication of the large-scale energy transfer.

The inverse cascade which arise from Eq. (5) is not intermittent. The probability distribution functions (pdfs) of the longitudinal velocity increments $\delta_r v = [\mathbf{v}(\mathbf{x} + \mathbf{r}) - \mathbf{v}(\mathbf{x})] \cdot \hat{\mathbf{r}}$ at distance r within the inertial range are self-similar and almost Gaussian (see inset of Fig. 4). The scaling of the second and the fourth order moment of velocity increments $S_2(r) = \langle (\delta_r v)^2 \rangle$; $S_4(r) = \langle (\delta_r v)^4 \rangle$ follow the dimensional scaling $S_p(r) \sim r^{p/3}$ (see Fig. 4). This is a signature of all known inverse cascades: when fluctuations are transferred from faster to slower degrees of freedom [27]. Previous studies have shown the possibility to produce large scale motion by non-parity invariant small-scales forcing only at small Reynolds numbers or in the quasi-linear regime [25]. Conversely, our results do not trivially originate from the projection of the forcing on the positive helicity states but is a genuine effect of the non-linear dynamics. To assess this issue we performed a test simulation of the complete NS equation with the same projected forcing used in Eq. (5). After an initial transient in which part of the energy accumulates at the forcing scale, a direct cascade sets in and all the energy injected is transferred toward small scales. This excludes the possibility that the forcing alone could be responsible for the inverse energy transfer observed in the decimated NS equation.

In conclusions, we have presented theoretical and numerical evidences that a *screwed* version of the NS equations, such that only modes with a given sign of helicity are retained, displays inverse energy transfer mechanism. This phenomenon, which has been previously observed

only in 2D turbulence or in strongly anisotropic 3D flows under bidimensionalization effects, is here observed for the first time in a fully isotropic 3D system and is intrinsically connected to the non-linear dynamics of *all* flows in nature.

The scientific impact of our findings is manifold. First, it allows to highlight those backward events in the energy transfer mechanism which are known to exist also in *un-truncated* NS equations and that are one of the main theoretical and applied challenges, see e.g. [24] for the case of sub-grid modelling in Large Eddy Simulations. Second, the link between backward energy events with the helical nature of triad interaction, shows the key role of the coupled energy-helicity dynamics. Third, by clearly detecting which triadic interaction is responsible for forward and backward energy transfer, we pave the road for closure and analytical approaches aimed at understanding the whole energy transfer distribution.

This study also opens the way to further investigations. An obvious extension would be to integrate Eq. (5) with a large scale forcing. In this case a pure forward helicity cascade must develop, provided that energy is removed at the forcing scale to avoid pile-up of fluctuations. More interesting, one could consider the case of a complementary decimation with respect to the one discussed here, i.e. eliminating only those triads that transfer energy backward. It is very tempting to speculate that such system could display a direct energy cascade with reduced –or even vanishing– intermittency, because one has removed all the *obstacles*, i.e. those events in which the forward energy transfer is stopped and/or reversed by the interaction with the helicity flux. Numerical simulations exploring these cases are ongoing and will be reported elsewhere. Finally, similar decomposition may shed lights also in the evolution of conducting fluids, where three invariants, kinetic plus magnetic energy, cross helicity and magnetic helicity are known to produce a reach phenomenology [28]. We acknowledge useful discussion with U. Frisch. L.B. acknowledge the kind hospitality from the Observatoire de la Cote d’Azur in Nice where part of this work was done. We acknowledge the European COST Action MP0806.

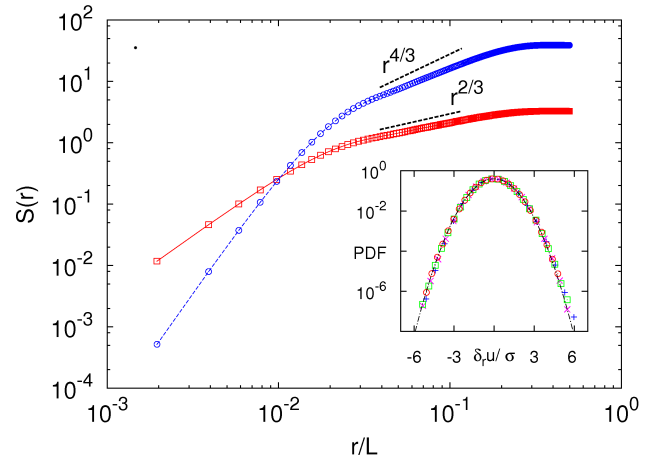


FIG. 4: (color online) Second order (squares) and fourth order (circles) structure functions in real space. Black solid and dashed lines represent the dimensional scaling $\propto r^{2/3}$ and $r^{4/3}$, respectively. Inset: pdfs of δu at various scale $r = [1/4, 1/8, 1/16, 1/32]L$, compared with the Gaussian distribution (dash-dotted line). Notice the perfect rescaling, supporting the absence of intermittency.

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